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LARGE DEFLECTION ANALYSIS OF PLATES ON ELASTIC FOUNDATION BY THE BOUNDARY ELEMENT METHOD

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Abstract—A boundary element method is developed for the large deflection analysis of thin elastic plates resting on elastic foundation. The subgrade reaction may depend linearly (Winkler-type) or nonlinearly on the deflection as well as on the point coordinates (nonhomogeneous subgrade). Moderately large deflections are examined as described by the von Kármán equations. The plate may have arbitrary shape and its boundary may be subjected to any type of boundary condition. The proposed method uses the fundamental solution of the linear plate theory and treats the nonlinearities as well as the subgrade reaction as unknown domain forces. Numerical results are presented to illustrate the method and demonstrate its effectiveness and accuracy.

1. INTRODUCTION

With the increased use of strong and light weight structures many problems of nonlinear deformations arise. In many technical fields flexible plates find use with deflections which are of the order of magnitude of the thickness of the plate but still small relative to the overall dimensions. Such for example are the fields of aircraft construction, shipbuilding, hydrospace, transportation, building and fluid tank construction. While a linear analysis often provides useful information in a structural problem, it can seldom provide insight into actual failures or the very many phenomena associated with nonlinear systems.

The governing equations for nonlinear behaviour of plates are those proposed by von Kármán (Voil'mir, 1967) which describe the behaviour of moderately large deflections. These equations are restricted to the condition that the shears, elongations and rotations are small compared to unity, but the rotations may be moderately large compared to elongations and shears. This geometry condition is common when the deflections are of the order of magnitude of the plate thickness (Novozhilov, 1953).

Analytic solution methods, exact or approximate, depend on the shape of the plate. Extensive literature on these methods can be found in Chia (1980). However, realistic problems can be solved by numerical methods. The finite difference method has been largely used. Nevertheless, this method becomes too sophisticated when the boundary does not conform with a coordinate system. The finite element method for large deflection of plates is well established (Oden, 1972). More recently, the boundary element method (BEM) has been developed to treat large deflections of plates (Kamiya and Sawaki, 1982; Tanaka, 1984; Ye and Liu, 1985; Nerantzaki and Katsikadelis, 1988; Katsikadelis and Nerantzaki, 1988).

The von Kármán equations can describe large deflections of plates on elastic foundation if they are augmented with the subgrade reaction term. The solution to this problem becomes more difficult even in the simplest case of Winkler-type subgrade reactions. To the author's knowledge, the existing approximate solutions and numerical results are restricted only to circular and rectangular plates on Winkler foundation (Sinha, 1963; Bolton, 1972; Datta, 1975).

For small deflections (linear analysis) of plates resting on elastic foundation the BEM has been well established by Katsikadelis and Armenakas (1984a,b), Katsikadelis and Kallivokas (1986, 1988), Costa and Brebbia (1985) and Bezine (1988). In this paper the BEM is developed to analyze plates on elastic foundation governed by moderately large deflections.

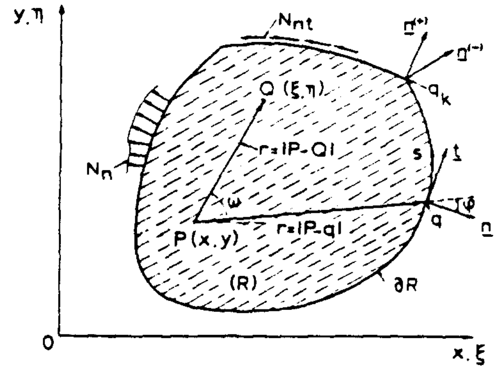

Fig. 1. Notations and coordinates.

The subgrade reaction may depend linearly or nonlinearly on the deflection as well as on the point coordinates (nonhomogeneous foundation). The proposed method uses the fundamental solution for the linear plate theory and treats the nonlinearities as well as the subgrade reaction as unknown domain forces. The resulting integral equations are solved numerically by developing an effective technique. Certain numerical examples are presented to illustrate the method and demonstrate its efficiency and the accuracy of the results.

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

Consider a thin elastic plate of thickness h, occupying a two-dimensional arbitrary shaped region R, bounded by a curve ∂R and resting on an elastic foundation with subgrade reaction p (Fig. 1). The nonlinear behaviour of the plate for moderately large deflections is governed by the differential equations proposed by von Kármán which in this case can be written as (Voil'mir, 1967)

$$\nabla^4 w = \frac{g}{D} - \frac{p}{D} + \frac{h}{D} L(w, F) \tag{1}$$

in R,

$$\nabla^4 F = -\frac{E}{2} L(w, w) \tag{2}$$

where $w = w(x, y)$ is the deflection function and $F(x, y)$ is an Airy-type stress function for the membrane stress; $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate having elastic constants E and ν; $g = g(x, y)$ is the transverse loading distributed on the plate; $p = p(x, y; w)$ is, in general, a nonlinear function of the point (x, y) and of the deflection w; and $L(w, F)$ is a nonlinear differential operator applied to w and F and represents

$$L(w, F) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\, \partial y}\frac{\partial^2 F}{\partial x\, \partial y}. \tag{3}$$

$L(w, w)$ is obtained by replacing F with w in eqn (3). $L(w, F)$ and $L(w, w)$ define the nonlinearity of the problem which is due to the coupling of the transverse deflection with the membrane deformation.

The plate is subjected to the following boundary conditions on the boundary ∂R of the plate;

$$\alpha_1 w + \alpha_2 V^* w = \alpha_3 \tag{4a}$$

$$\beta_1 \frac{\partial w}{\partial n} + \beta_2 M w = \beta_3 \tag{4b}$$

$$F = \gamma_1 \tag{5a}$$

$$\frac{\partial F}{\partial n} = \gamma_2, \tag{5b}$$

where $\alpha_i = \alpha_i(s)$, $\beta_i = \beta_i(s)$ $(i = 1, 2, 3)$ and $\gamma_k = \gamma_k(s)$ $(k = 1, 2)$ are functions specified on ∂R. V^*w and Mw are the reacting transverse force and the bending moment along the boundary and they are given as

$$V^*w = Vw + N_n \frac{\partial w}{\partial n} + N_{nt} \frac{\partial w}{\partial t} \tag{6a}$$

$$Mw = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2}\right), \tag{6b}$$

where

$$Vw = -D\left[\frac{\partial}{\partial n}\nabla^2 w - (\nu - 1)\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial n\,\partial t}\right)\right] \tag{7}$$

is the effective shearing force of the linear theory. The rest terms in eqn (6a) are due to the contribution of the membrane force components N_n and N_{nt} in the transverse direction (Dym and Shames, 1973). Using intrinsic coordinates and noting that (Katsikadelis, 1982)

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial s}, \quad \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial s^2} + \kappa \frac{\partial w}{\partial n}, \quad \frac{\partial^2 w}{\partial n\,\partial t} = \frac{\partial^2 w}{\partial n\,\partial s} - \kappa \frac{\partial w}{\partial s},$$

eqns (6a,b) are written as follows:

$$V^*w = -D\left[\Psi - (\nu - 1)\frac{\partial}{\partial s}\left(\frac{\partial X}{\partial s} - \kappa \frac{\partial \Omega}{\partial s}\right)\right] + N_n X + N_{nt} \frac{\partial \Omega}{\partial s} \tag{8a}$$

$$Mw = -D\left[\Phi + (\nu - 1)\left(\frac{\partial^2 \Omega}{\partial s^2} + \kappa X\right)\right], \tag{8b}$$

where the following notation has been used

$$\Omega = w, \quad X = \frac{\partial w}{\partial n}, \quad \Phi = \nabla^2 w, \quad \Psi = \frac{\partial}{\partial n}\nabla^2 w \quad \text{and} \quad \kappa = \kappa(s) \tag{9}$$

is the curvature of the boundary. Expressions (8a,b) for Mw and V^*w, are convenient to treat the case of nonvanishing prescribed membrane edge forces. It is apparent that all kinds of boundary conditions with respect to the transverse deflection w (clamped edge, simply supported edge, free edge, elastically supported edge, mixed boundary conditions) may be treated by specifying appropriately the functions $\alpha_i(s)$, $\beta_i(s)$. On the other hand, stress boundary conditions [eqns (5a,b)] are considered for the membrane stresses which are rather more easily expressible in terms of the stress function (case of movable edge subjected to prescribed inplane edge forces).

3. INTEGRAL EQUATION FORMULATION

For any function $u(x, y)$ which is the solution of the biharmonic equation

$$\nabla^4 u = g, \tag{10}$$

the following integral representation may be obtained (Katsikadelis and Armenakas, 1984a)

$$u(P) = \iint_R vg\,\mathrm{d}\sigma - \int_{\partial R}\left(v\frac{\partial}{\partial n}\nabla^2 u - u\frac{\partial}{\partial n}\nabla^2 v - \frac{\partial v}{\partial n}\nabla^2 u + \frac{\partial u}{\partial n}\nabla^2 v\right)\mathrm{d}s, \tag{11}$$

where

$$v = \frac{1}{8\pi}r^2 \ln r, \quad r = |P-Q|, \quad P, Q \in R \tag{12}$$

is the fundamental solution to eqn (10).

Applying the Laplacian operator to eqn (11) and taking into account that it is $\nabla^4 v = \nabla^2((\partial/\partial n)\nabla^2 v) = 0$ in the boundary integrals, we obtain

$$\nabla^2 u(P) = \iint_R \nabla^2 vg\,\mathrm{d}\sigma - \int_{\partial R}\left(\nabla^2 v\frac{\partial}{\partial n}\nabla^2 u - \frac{\partial}{\partial n}\nabla^2 v\nabla^2 u\right)\mathrm{d}s. \tag{13}$$

Applying eqns (11) and (13) to the functions w and F satisfying eqns (1) and (2), respectively, we obtain the following integral representations

$$w(P) = \frac{1}{2\pi D}\iint_R \Lambda_4(r)g\,\mathrm{d}\sigma - \frac{1}{2\pi D}\iint_R \Lambda_4(r)p(w)\,\mathrm{d}\sigma + \frac{h}{2\pi D}\iint_R \Lambda_4(r)L(w,F)\,\mathrm{d}\sigma$$
$$- \frac{1}{2\pi}\int_{\partial R}[\Lambda_1(r)\Omega + \Lambda_2(r)X + \Lambda_3(r)\Phi + \Lambda_4(r)\Psi]\,\mathrm{d}s \tag{14}$$

$$F(P) = -\frac{E}{4\pi}\iint_R \Lambda_4(r)L(w,w)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R}[\Lambda_1(r)\bar{\Omega} + \Lambda_2(r)\bar{X} + \Lambda_3(r)\bar{\Phi} + \Lambda_4(r)\bar{\Psi}]\,\mathrm{d}s \tag{15}$$

$$\nabla^2 w(P) = \frac{1}{2\pi D}\iint_R \Lambda_2(r)g\,\mathrm{d}\sigma - \frac{1}{2\pi D}\iint_R \Lambda_2(r)p(w)\,\mathrm{d}\sigma + \frac{h}{2\pi D}\iint_R \Lambda_2(r)L(w,F)\,\mathrm{d}\sigma$$
$$- \frac{1}{2\pi}\int_{\partial R}[\Lambda_1(r)\Phi + \Lambda_2(r)\Psi]\,\mathrm{d}s \tag{16}$$

$$\nabla^2 F(P) = -\frac{E}{4\pi}\iint_R \Lambda_2(r)L(w,w)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R}[\Lambda_1(r)\bar{\Phi} + \Lambda_2(r)\bar{\Psi}]\,\mathrm{d}s, \tag{17}$$

in which Ω, X, Φ, Ψ are defined by eqns (9) and $\bar{\Omega}$, $\bar{X}$, $\bar{\Phi}$, $\bar{\Psi}$ denote

$$\bar{\Omega} = F, \quad \bar{X} = \frac{\partial F}{\partial n}, \quad \bar{\Phi} = \nabla^2 F, \quad \bar{\Psi} = \frac{\partial}{\partial n}\nabla^2 F. \tag{18}$$

Moreover, the kernels $\Lambda_i(r)$ $(i = 1, 2, 3, 4)$ are given by the following relations

$$\Lambda_1(r) = -2\pi\frac{\partial}{\partial n}\nabla^2 v = -\frac{\cos\varphi}{r} \tag{19a}$$

$$\Lambda_2(r) = 2\pi\nabla^2 v = \ln r + 1 \tag{19b}$$

$$\Lambda_3(r) = -2\pi\frac{\partial v}{\partial n} = -\tfrac{1}{4}(2r\ln r + r)\cos\varphi \tag{19c}$$

$$\Lambda_4(r) = 2\pi v = \tfrac{1}{4}r^2 \ln r, \tag{19d}$$

where φ is the angle between the directions of $\mathbf{r}$ and $\mathbf{n}$ (see Fig. 1).

Equations (14) and (15) can give the solution of the problem if the eight unknown boundary quantities Ω, X, Φ, Ψ, $\bar{\Omega}$, $\bar{X}$, $\bar{\Phi}$ and $\bar{\Psi}$ involved in the boundary integrals as well as the subgrade reaction $p(w)$ and the six derivatives $w_{,xx}$, $w_{,xy}$, $w_{,yy}$, $F_{,xx}$, $F_{,xy}$, $F_{,yy}$ involved in the domain integrals, are first established. We obtain the necessary equations for the determination of these unknown quantities by working as follows.

Using the procedure presented by Katsikadelis and Armenakas (1984a), two boundary integral equations are obtained for the function w from eqns (14) and (16). Thus, letting point P coincide with a point $p \in \partial R$ in eqn (14) and in its Laplacian, eqn (16), and noting that the kernel $\Lambda_1(r)$ behaves like a double layer potential, we obtain the boundary integral equations

$$\pi\Omega = \frac{1}{D}\iint_R \Lambda_4 g \,\mathrm{d}\sigma - \frac{1}{D}\iint_R \Lambda_4 p \,\mathrm{d}\sigma + \frac{h}{D}\iint_R \Lambda_4 L(w, F)\,\mathrm{d}\sigma - \int_{\partial R} (\Lambda_1\Omega + \Lambda_2 X + \Lambda_3\Phi + \Lambda_4\Psi)\,\mathrm{d}s \tag{20a}$$

$$\pi\Phi = \frac{1}{D}\iint_R \Lambda_2 g \,\mathrm{d}\sigma - \frac{1}{D}\iint_R \Lambda_2 p \,\mathrm{d}\sigma + \frac{h}{D}\iint_R \Lambda_2 L(w, F)\,\mathrm{d}\sigma - \int_{\partial R} (\Lambda_1\Phi + \Lambda_2\Psi)\,\mathrm{d}s. \tag{20b}$$

Moreover, by differentiating eqn (14) twice with respect to x and y, the integral representations of the functions $w_{,xx}$, $w_{,yy}$, $w_{,xy}$ are obtained. Thus, we have

$$w_{,xx}(P) = \frac{1}{2\pi D}\iint_R (\Lambda_4)_{,xx} g \,\mathrm{d}\sigma - \frac{1}{2\pi D}\iint_R (\Lambda_4)_{,xx} p \,\mathrm{d}\sigma + \frac{h}{2\pi D}\iint_R (\Lambda_4)_{,xx} L(w, F)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R} [(\Lambda_1)_{,xx}\Omega + (\Lambda_2)_{,xx}X + (\Lambda_3)_{,xx}\Phi + (\Lambda_4)_{,xx}\Psi]\,\mathrm{d}s \tag{20c}$$

$$w_{,xy}(P) = \frac{1}{2\pi D}\iint_R (\Lambda_4)_{,xy} g \,\mathrm{d}\sigma - \frac{1}{2\pi D}\iint_R (\Lambda_4)_{,xy} p \,\mathrm{d}\sigma + \frac{h}{2\pi D}\iint_R (\Lambda_4)_{,xy} L(w, F)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R} [(\Lambda_1)_{,xy}\Omega + (\Lambda_2)_{,xy}X + (\Lambda_3)_{,xy}\Phi + (\Lambda_4)_{,xy}\Psi]\,\mathrm{d}s \tag{20d}$$

$$w_{,yy}(P) = \frac{1}{2\pi D}\iint_R (\Lambda_4)_{,yy} g \,\mathrm{d}\sigma - \frac{1}{2\pi D}\iint_R (\Lambda_4)_{,yy} p \,\mathrm{d}\sigma + \frac{h}{2\pi D}\iint_R (\Lambda_4)_{,yy} L(w, F)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R} [(\Lambda_1)_{,yy}\Omega + (\Lambda_2)_{,yy}X + (\Lambda_3)_{,yy}\Phi + (\Lambda_4)_{,yy}\Psi]\,\mathrm{d}s. \tag{20e}$$

Similarly, using the integral representations (15) and (17), we obtain for function F

$$\pi\bar{\Omega} = -\frac{E}{2}\iint_R \Lambda_4 L(w, w)\,\mathrm{d}\sigma - \int_{\partial R} (\Lambda_1\bar{\Omega} + \Lambda_2\bar{X} + \Lambda_3\bar{\Phi} + \Lambda_4\bar{\Psi})\,\mathrm{d}s \tag{21a}$$

$$\pi\bar{\Phi} = -\frac{E}{2}\iint_R \Lambda_2 L(w, w)\,\mathrm{d}\sigma - \int_{\partial R} (\Lambda_1\bar{\Phi} + \Lambda_2\bar{\Psi})\,\mathrm{d}s \tag{21b}$$

$$F_{xx}(P) = -\frac{E}{4\pi}\iint_R (\Lambda_4)_{xx} L(w, w)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R} [(\Lambda_1)_{xx}\bar{\Omega} + (\Lambda_2)_{xx}\bar{X} + (\Lambda_3)_{xx}\Phi + (\Lambda_4)_{xx}\Psi]\,\mathrm{d}s \tag{21c}$$

$$F_{xy}(P) = -\frac{E}{4\pi}\iint_R (\Lambda_4)_{xy} L(w, w)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R} [(\Lambda_1)_{xy}\bar{\Omega} + (\Lambda_2)_{xy}\bar{X} + (\Lambda_3)_{xy}\Phi + (\Lambda_4)_{xy}\Psi]\,\mathrm{d}s \tag{21d}$$

$$F_{yy}(P) = -\frac{E}{4\pi}\iint_R (\Lambda_4)_{yy} L(w, w)\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\partial R} [(\Lambda_1)_{yy}\bar{\Omega} + (\Lambda_2)_{yy}\bar{X} + (\Lambda_3)_{yy}\Phi + (\Lambda_4)_{yy}\Psi]\,\mathrm{d}s, \tag{21e}$$

where

$$(\Lambda_1)_{xx} = -(\Lambda_1)_{yy} = -\frac{2\cos(2\omega-\varphi)}{r^3}, \quad (\Lambda_1)_{xy} = -\frac{2\sin(2\omega-\phi)}{r^3},$$

$$(\Lambda_2)_{xx} = -(\Lambda_2)_{yy} = -\frac{\cos 2\omega}{r^2}, \quad (\Lambda_2)_{xy} = -\frac{\sin 2\omega}{r^2},$$

$$(\Lambda_3)_{xx} = \frac{\sin 2\omega \sin\varphi - \cos\varphi}{2r}, \quad (\Lambda_3)_{yy} = \frac{-\sin 2\omega \sin\varphi - \cos\varphi}{2r},$$

$$(\Lambda_3)_{xy} = -\frac{\cos 2\omega \sin\varphi}{2r}, \quad (\Lambda_4)_{xx} = \frac{2\ln r + 2 + \cos 2\omega}{4},$$

$$(\Lambda_4)_{yy} = \frac{2\ln r + 2 - \cos 2\omega}{4}, \quad (\Lambda_4)_{xy} = \frac{\sin 2\omega}{4} \tag{22}$$

and ω is the angle between the x axis and r (see Fig. 1).

Furthermore, using notations (9) and (18) and eqns (8a,b), the boundary conditions (4a,b) and (5a,b) are written as

$$\alpha_1\Omega - D\alpha_2\left[\Psi - (\nu-1)\frac{\partial}{\partial s}\left(\frac{\partial X}{\partial s} - \kappa\frac{\partial\Omega}{\partial s}\right)\right] + \alpha_2\left(N_n X + N_{nt}\frac{\partial\Omega}{\partial s}\right) = \alpha_3 \tag{23a}$$

$$\beta_1 X - D\beta_2\left[\Phi + (\nu-1)\left(\frac{\partial^2\Omega}{\partial s^2} + \kappa X\right)\right] = \beta_3 \tag{23b}$$

$$\bar{\Omega} = \gamma_1 \quad \bar{X} = \gamma_2. \tag{23c,d}$$

Relations (14), (20), (21) and (23) provide the necessary equations to establish the unknown quantities. Equations (23c,d) are directly eliminated from eqns (21) and the unknown quantities are reduced to 13. However, a further reduction of the unknowns is not possible, because, in a general case, substitution of eqns (23a,b) into eqns (20) yields integro-differential equations whose solution involves considerable difficulties. For this reason, it is more convenient to treat eqns (23a,b) as boundary differential equations using the new boundary differential integral equation method developed by Katsikadelis and Armenakas (1989). Thus, eqns (14), (20), (21) and (23a,b) are considered as a system of 13 simultaneous equations, from which the unknown quantities can be established.

The bending moments, the twisting moment, and the membrane forces are obtained by first computing the derivatives from eqns (20) and (21) and, subsequently, using the relations

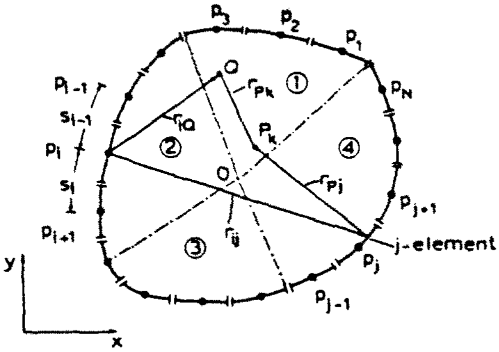

Fig. 2. Boundary discretization (N boundary elements) and domain discretization (four finite sectors).

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \quad M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}$$

$$N_x = h\frac{\partial^2 F}{\partial y^2}, \qquad N_y = h\frac{\partial^2 F}{\partial x^2}, \qquad N_{xy} = -h\frac{\partial^2 F}{\partial x\,\partial y}. \tag{24}$$

4. SOLUTION PROCEDURE

An analytical solution of the system of eqns (14), (20), (21) and (23a,b) is out of question. However, a numerical solution is feasible. Thus, the boundary integrals can be approximated using BEM with N constant elements and parabolic approximation of the curved boundary elements, while the derivatives in the boundary differential equations [(23a,b)] can be approximated by unevenly spaced finite difference schemes. Concerning the domain integrals involving the unknown second derivatives and the subgrade reaction, they can be evaluated using M-point Gauss integration over domains of arbitrary shape (Nerantzaki and Katsikadelis, 1988; Katsikadelis, 1991; see also the Appendix). Subsequently, application of the collocation technique at the N boundary nodal points and at the M Gauss integration points inside the domain R (see Fig. 2) yields the necessary equations for the unknown values of the boundary and domain quantities.

Thus, using the procedure described previously and grouping the discretized equations appropriately, we obtain the following nonlinear set of algebraic equations

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{0} & \mathbf{A}_{34} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{X} \\ \mathbf{\Phi} \\ \mathbf{\Psi} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}[\mathbf{f}] + \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}[\mathbf{p}] \tag{25a}$$

$$[\mathbf{w}] = [\mathbf{B}_5] + [\mathbf{C}_5][\mathbf{f}] + [\mathbf{D}_5][\mathbf{p}] + [\mathbf{A}_{51} \quad \mathbf{A}_{52} \quad \mathbf{A}_{53} \quad \mathbf{A}_{54}] \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{X} \\ \mathbf{\Phi} \\ \mathbf{\Psi} \end{bmatrix} \tag{25b}$$

$$\begin{bmatrix} \mathbf{w}_{,xx} \\ \mathbf{w}_{,xy} \\ \mathbf{w}_{,yy} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_6 \\ \mathbf{B}_7 \\ \mathbf{B}_8 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_6 \\ \mathbf{C}_7 \\ \mathbf{C}_8 \end{bmatrix}[\mathbf{f}] + \begin{bmatrix} \mathbf{D}_6 \\ \mathbf{D}_7 \\ \mathbf{D}_8 \end{bmatrix}[\mathbf{p}] + \begin{bmatrix} \mathbf{A}_{61} & \mathbf{A}_{62} & \mathbf{A}_{63} & \mathbf{A}_{64} \\ \mathbf{A}_{71} & \mathbf{A}_{72} & \mathbf{A}_{73} & \mathbf{A}_{74} \\ \mathbf{A}_{81} & \mathbf{A}_{82} & \mathbf{A}_{83} & \mathbf{A}_{84} \end{bmatrix} \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{X} \\ \mathbf{\Phi} \\ \mathbf{\Psi} \end{bmatrix} \tag{25c}$$

$$\begin{bmatrix} \bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi} \\ \mathbf{\Psi} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{B}}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{C}}_1 \\ \bar{\mathbf{C}}_2 \end{bmatrix}[\bar{\mathbf{f}}] \tag{25d}$$

$$\begin{bmatrix} \mathbf{F}_{xx} \\ \mathbf{F}_{xy} \\ \mathbf{F}_{yy} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{B}}_3 \\ \bar{\mathbf{B}}_4 \\ \bar{\mathbf{B}}_5 \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{C}}_3 \\ \bar{\mathbf{C}}_4 \\ \bar{\mathbf{C}}_5 \end{bmatrix} [\bar{\mathbf{f}}] + \begin{bmatrix} \bar{\mathbf{A}}_{33} & \bar{\mathbf{A}}_{34} \\ \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{44} \\ \bar{\mathbf{A}}_{53} & \bar{\mathbf{A}}_{54} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}} \\ \bar{\mathbf{\Psi}} \end{bmatrix}, \tag{25e}$$

where

$[\mathbf{w}]$ is an $M\times 1$ vector of the unknown deflections at the M Gauss points inside R; $[\mathbf{w}_{xx}\ \ \mathbf{w}_{xy}\ \ \mathbf{w}_{yy}]$, $[\mathbf{F}_{xx}\ \ \mathbf{F}_{xy}\ \ \mathbf{F}_{yy}]$ are $3M\times 1$ vectors of the unknown values of second derivatives at the M Gauss points; $\mathbf{\Omega}$, $\mathbf{X}$, $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\bar{\mathbf{\Phi}}$, $\bar{\mathbf{\Psi}}$ are $N\times 1$ vectors of the unknown values of the boundary quantities at the N boundary nodal points; $[\mathbf{f}]$, $[\bar{\mathbf{f}}]$, $[\mathbf{p}]$ are $M\times 1$ vectors of the unknown values of the functions $f = w_{,xx}f_{,yy} + w_{,yy}f_{,xx} - 2w_{,xy}f_{,xy}$, $\bar{f} = 2(w_{,xx}w_{,yy} - w_{,xy}^2)$ and $p = p(w)$ at the Gauss points; $\mathbf{A}_{\alpha\beta}$, $\bar{\mathbf{A}}_{\alpha\beta}$, $\mathbf{B}_\alpha$, $\bar{\mathbf{B}}_\alpha$, $\mathbf{C}_\alpha$, $\bar{\mathbf{C}}_\alpha$, $\mathbf{D}_\alpha$ are constant matrices.

Equations (25a) are linear with respect to the boundary quantities $\mathbf{\Omega}$, $\mathbf{X}$, $\mathbf{\Phi}$, $\mathbf{\Psi}$ and they can be solved for them and substituted into eqns (25b) and (25c). Similarly, eqn (25d) can be solved for the boundary quantities $\bar{\mathbf{\Phi}}$, $\bar{\mathbf{\Psi}}$ and substituted into eqn (25e). These eliminations yield the following nonlinear system of algebraic equations:

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{U} \\ \bar{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \mathbf{0} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{p}(\mathbf{w}) \\ \mathbf{f}(\mathbf{U}, \bar{\mathbf{U}}) \\ \bar{\mathbf{f}}(\mathbf{U}) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \mathbf{G}_3 \end{bmatrix}, \tag{26}$$

where $\mathbf{w}$, $\mathbf{U}$ and $\bar{\mathbf{U}}$ are the unknown vectors of the values of the deflections w_i, the derivatives $(w_{,xx})_i$, $(w_{,xy})_i$, $(w_{,yy})_i$ and $(F_{,xx})_i$, $(F_{,xy})_i$, $(F_{,yy})_i$, respectively, at the M Gauss points within the domain of the plate; $\mathbf{H}_{ij}$ are constant matrices and G_i are constant vectors. Finally, $\mathbf{p}(\mathbf{w})$, $\mathbf{f}(\mathbf{U}, \bar{\mathbf{U}})$ and $\bar{\mathbf{f}}(\mathbf{U})$ are the vectors of the values of the nonlinear functions $p(w)$, $L(w, F)$, and $L(w, w)$ at the Gauss points. In evaluating the constant matrices in eqns (25) certain singular domain integrals occur, which are computed using the technique presented in Katsikadelis and Nerantzaki (1988).

Equations (26) are solved iteratively by step increasing the loading to yield the values of $w, w_{,xx}, \ldots, F_{,yy}$ at the Gauss points. Backsubstitution into eqns (25) gives the unknown boundary quantities at the nodal boundary points. Subsequently, the deflection and stress functions are computed at any point $P \in R$ using the discretized form of eqns (14) and (15).

5. NUMERICAL RESULTS

On the basis of the analysis and the numerical procedure presented in the previous sections, a computer program has been written and numerical results have been obtained. Since the main purpose of this paper is to present the basic principles of the proposed method and demonstrate its efficiency, the obtained numerical results are limited to a circular plate with two different types of boundary condition, for which numerical results are also available for comparison. In all the examples treated, the numerical results have been obtained using 60 constant boundary elements and 100 Gauss nodal points by dividing the interior of the plate into four sectors on each of which a 25-point Gauss–Radau integration is performed. For the iterative solution of eqns (26) about 250 iterations were needed at each load step $\Delta q = 1$ and for a convergence equal to 10^{-4} criterion. The example problems treated are the following:

(1) A uniformly loaded circular plate with radius a and clamped movable (CM) edge (i.e. $w = \partial w/\partial n = F = \partial F/\partial n = 0$ on the boundary) resting on a Winkler-type elastic foundation. The numerical results have been obtained for $a/h = 50$, $\nu = 0.30$ and are

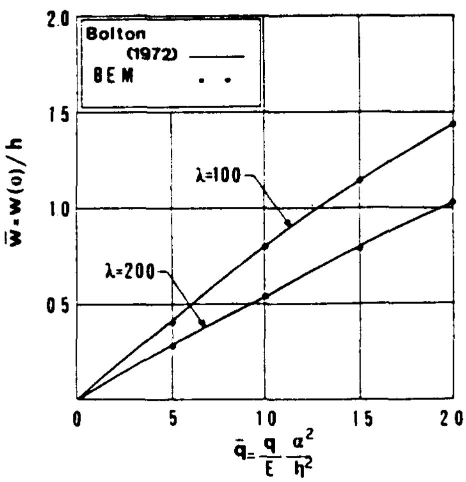

Fig. 3. Central deflection $\bar{w}$ versus the load $\bar{q}$ in a CM circular plate ($\nu = 0.30$) resting on Winkler-type elastic foundation.

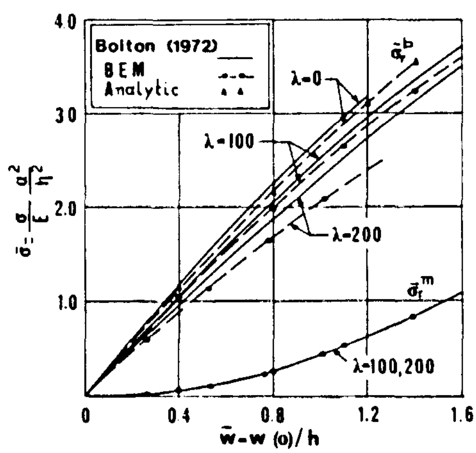

Fig. 4. Central membrane ($\bar{\sigma}_r^m$) and bending ($\bar{\sigma}_r^b$) stresses versus central deflection $\bar{w}$ in a CM circular plate ($\nu = 0.3$) resting on a Winkler-type elastic foundation ($\lambda = 100, 200$).

Table 1. Deflections, bending and membrane stresses along the radius in a uniformly loaded CM circular plate resting on a Winkler-type elastic foundation ($\nu = 0.3$, $\bar{q} = 15$, $\lambda = 100$)

r/a	$\bar{w}$	$\bar{\sigma}_r^b$	$\bar{\sigma}_t^b$	$\bar{\sigma}_r^m$	$\bar{\sigma}_t^m$
0.098	1.108	2.547	2.553	0.536	0.521
0.304	0.961	2.366	2.466	0.470	0.329
0.562	0.592	1.277	1.936	0.303	−0.113
0.800	0.179	−1.820	0.403	0.127	−0.478
0.960	0.009	−5.490	−1.417	0.040	−0.468

presented in Figs 3 and 4 and Table 1. In Fig. 3 the central deflection $\bar{w} = w/h$ is plotted versus the load $\bar{q} = qa^4/Eh^4$ for two values of the subgrade reaction parameter $\lambda = ka^4/D$ of the Winkler-type elastic foundation, $p = kw$, k being the constant subgrade modulus. From this figure it is seen that the obtained results are in very good agreement with those obtained by Bolton (1972). However, as shown in Fig. 4, significant deviations are observed in the radial bending stress $\bar{\sigma}_r^b = \sigma_r^b a^2/Eh^2$ ($\sigma_r^b = 6M_r/h^2$) between the results given by Bolton (1972) and the corresponding ones obtained by the proposed method. The analytical results have been obtained using a series solution (Timoshenko and Woinowsky-Krieger, 1959). The differences become larger as λ and $\bar{q}$ increase.

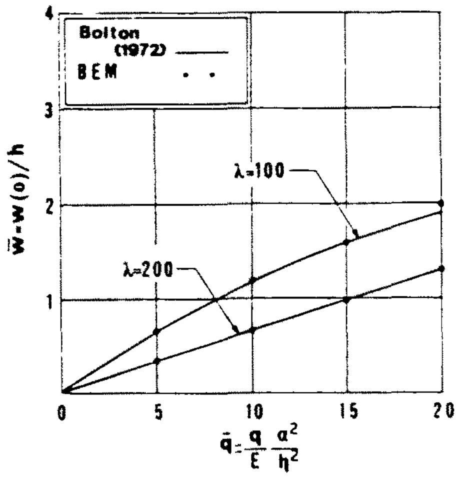

Fig. 5. Central deflection $\bar{w}$ versus the load $\bar{q}$ in a SM circular plate ($\nu = 0.3$) resting on a Winkler-type elastic foundation.

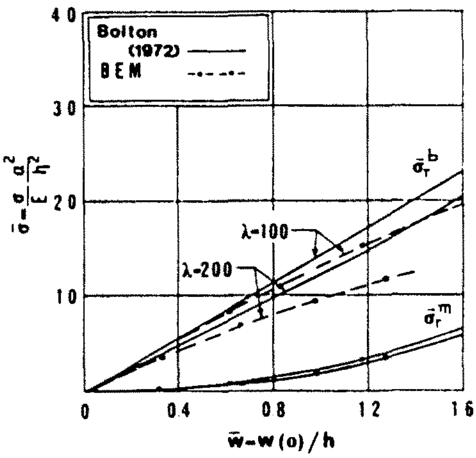

Fig. 6. Central membrane ($\bar{\sigma}_r^m$) and bending ($\bar{\sigma}_r^b$) stresses versus central deflection $\bar{w}$ in a SM circular plate ($\nu = 0.3$) resting on a Winkler-type elastic foundation ($\lambda = 100, 200$).

Since for $\lambda \neq 0$ there is no exact solution, one cannot tell with certainty which results are more accurate. Nevertheless, there are two reasons which indicate that our results are more accurate. (a) For the special case $\lambda = 0$, the results obtained herein are in full agreement with those obtained by an analytic series solution (Nerantzaki and Katsikadelis, 1988; Timoshenko and Woinowsky-Krieger, 1959). (b) In our procedure, the curvature components $w_{,xx}$, $w_{,xy}$ and $w_{,yy}$ of the deflection are first established and subsequently are used to compute deflections which coincide with those given by Bolton (1972). With regards to the radial membrane stress $\bar{\sigma}_r^m = \sigma_r^m a^2/Eh^2$ ($\sigma_r^m = N_r/h$) it is seen from the same figure, that it varies negligibly with λ. The obtained results are in good agreement with those given by Bolton (1972). Finally, in Table 1, the variation of the deflection $\bar{w}$, radial stresses $\bar{\sigma}_r^b$ and $\bar{\sigma}_r^m$ and tangential stresses $\bar{\sigma}_t^b$ and $\bar{\sigma}_t^m$ along the radius, for $\bar{q} = 15$ are presented. These results are new and thus, their accuracy cannot be checked.

(2) A uniformly loaded circular plate with radius a and simply supported movable (SM) edge (i.e. $w = Mw = F = \partial F/\partial n = 0$ on the boundary) resting on a Winkler-type elastic foundation. The results have been obtained for $a/h = 50$, $\nu = 0.30$ and are presented in Figs 5 and 6 along with those given by Bolton (1972). As it is seen from these figures, there is very good agreement in the deflection and the radial membrane stress $\bar{\sigma}_r^m$.

However, the deviation in the radial bending stress, as in the case of a CM plate, is not negligible.

6. CONCLUDING REMARKS

An efficient boundary element approach is developed for the nonlinear analysis of thin elastic plates resting on elastic foundation. The method is not actually a pure boundary element method, since it also requires domain discretization to compute unknown quantities in the interior of the plate. However, the linear equations are still defined by the boundary discretization and thus, the method retains most of the advantages over the domain-type methods. It is worth mentioning that:

(a) The proposed method is suitable for analyzing plates having arbitrary shape and subjected to any type of boundary condition.
(b) The subgrade reaction may depend linearly or nonlinearly on the deflection.
(c) The method is well-suited for computer-aided analysis.
(d) Accurate results are obtained using a relatively small number of boundary and domain nodal points.

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APPENDIX

A Gaussian quadrature technique for regions of arbitrary shape

We review here the finite sector method (FSM) for the evaluation of domain integrals over regions of arbitrary shape.

Consider the integral

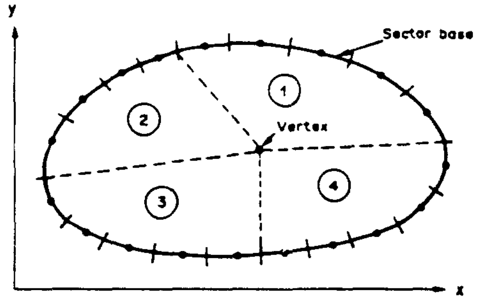

Fig. A1. Two-dimensional domain divided into four sectors.

$$\iint_R g(Q)\,\mathrm{d}\sigma_Q, \tag{A1}$$

where R is a two-dimensional region of arbitrary shape. We divide the region into a finite number of sectors by straight lines emanating from a point inside the region (common vertex of sectors) and reaching the boundary (see Fig. A1). For domains with complex geometry more than one vertex may be used. Subsequently, each sector is mapped onto a triangle on which a ready-to-use Gauss Radau integration scheme is employed. Thus, the integral (A1) may be written as

$$\iint_R g(Q)\,\mathrm{d}\sigma_Q = \sum_{k=1}^{S} \iint_{R_k^*} g(Q)|J_k(Q)|\,\mathrm{d}\sigma_Q = \sum_{k=1}^{S} \sum_{j=1}^{m} C_j^k g(Q_j^k)|J_k(Q_j^k)|, \tag{A2}$$

where S is the number of the sectors; m is the number of Gauss Radau points in the kth sector; C_j^k, Q_j^k ($j = 1, 2, \ldots, m$) are the weight factors and the Gauss Radau points in the kth sector; and $J_k(Q_j^k)$ are the values of the Jacobian of the transformation which transforms the kth sector onto the triangle R_k^*.

The transformation that maps the sector onto a triangle is given by Katsikadelis (1991) as

$$\bar{x} = f\left(\lambda\frac{\eta}{\xi}\right)\frac{\xi}{\lambda}, \quad \bar{y} = \eta, \tag{A3}$$

where $\bar{x} = f(\bar{y})$ is the equation of the sector base in local coordinates (see Fig. A2).

When this technique is used with BEM, the sector base consists of a group of consecutive boundary elements and it is convenient to approximate the function $\bar{x} = f(\bar{y})$ by an interpolating polynomial rather than using the analytic expression of the curve. Thus, if a polynomial approximation is used, the sector base is chosen so that it can be represented as

$$\bar{x} = f(\bar{y}) = \alpha_0 + \alpha_1\bar{y} + \alpha_2\bar{y}^2 + \cdots + \alpha_n\bar{y}^n. \tag{A4}$$

The coefficients α_i ($i = 0, 1, 2, \ldots, n$) are computed from the coordinates of the nodal and/or the extreme points of the boundary elements.

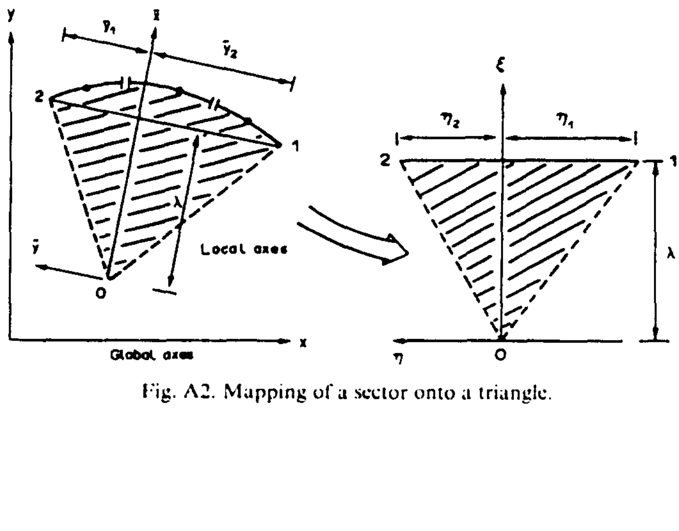

Fig. A2. Mapping of a sector onto a triangle.